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ADDENDUM

The BRST operator of U_q (sl(2)) and real forms

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Abstract. In a previous letter a nilpotent BRST operator Q_q was constructed for the q-deformation of the universal enveloping algebra of sl(2). The present addendum resolves issues of non-uniqueness in the construction of the BRST operator by imposing additional structure associated with real forms, and corrects a misprint in the previous work.

The question of whether q-deformed symmetries can be used in constructing theories with local gauge invariance is naturally treated in the framework of non-commutative differential geometry [1]. However, the view that the form of an eventual q-gauge invariant theory should be essentially determined by the constraint structure associated with quantization suggests that generalizations of the geometrical structure of BRST invariance should also provide insights [2]. From this perspective the work of Kunz *et al* [3] has provided an algebraic realization of the Woronowicz [4] cohomology+.

An independent study of a q-deformed BRST cohomology was given in [5], which forms the basis of the present comment. Assuming a quantum group symmetry to lead to a (local) deformed constraint algebra*, the construction considered a zero-dimensional analogue for the case of three constraints, adopting the structure of the Drinfeld [6] $\mathrm{sl}_q(2)$ algebra. A *nilpotent* BRST operator Q_q was constructed, an associated operator R_q such that $\{Q_q, R_q\} = C_q$ was introduced, with C_q the q-deformed Casimir invariant, and the resulting BRST cohomology of $U_q(sl(2))$ representations discussed.

As pointed out subsequently by Dayi [10], and illustrated with several examples, the q-BRST construction is not unique because of the arbitrariness in choices of constraint, all of which tend to the usual sl(2) algebra in the limit $q \to 1$. The present addendum provides a more complete discussion of the non-uniqueness of our algebraic construction of the BRST operator in the q-deformed case. The freedom of choice of constraints is in fact resolved if further algebraic structure is imposed via the existence of an adjoint operator associated with a real form of the algebra. In the light of this analysis we make further comments below on the work of Dayi [10], and also correct an error in our previous letter (cf (13) below and (9) of [5]).

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+ For other approaches to q-deformed quantization see for example Fei and Guo [7] and Majid [8].
* For an approach based on a loop space formulation see Miallet and Nijhoff [9].

For completeness a brief recapitulation of the algebraic framework (cf [5]) is given. Specifically it is assumed for $\mathrm{sl}_q(2)$ that the constraint algebra is generated by e, f and, in the most general case to be treated here, an arbitrary function $A(h)$, where [6]

$$\begin{aligned}
&[e, f] = [h]_q \\
&[h, e] = 2e \\
&[h, f] = -2f
\end{aligned} \tag{1}$$

with

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Correspondingly to e, f and h are associated with the ghost generators c^+, c^- and c^0 respectively, which satisfy the *standard* fermionic anticommutation relations with their antighost counterparts $\{c, \bar{c}\} = 1$. In this respect the treatment differs from that of [3] (where q-deformed ghosts are required as part of the geometrical formalism), but is consistent with [10].

Let the BRST operator Q_q be of the general form

$$\begin{aligned}
Q_q = ec^- + fc^+ + A(h)c^0 + B(h)\bar{c}^-c^-c^0 + C(h)\bar{c}^+c^+c^0 \\
+F(h)\bar{c}^0c^+c^- + E(h)\bar{c}^+\bar{c}^-c^+c^-c^0.
\end{aligned} \tag{2}$$

involving arbitrary functions A, B, C, F and E. Correspondingly the ghost number -1 counterpart R_q is taken as

$$\begin{aligned}
R_q = e\bar{c}^+ + f\bar{c}^- + J(h)\bar{c}^0 + K(h)\bar{c}^0\bar{c}^-c^- \\
+M(h)\bar{c}^0\bar{c}^+c^+ + N(h)\bar{c}^+\bar{c}^-c^0 + L(h)\bar{c}^+\bar{c}^-\bar{c}^0c^+c^-
\end{aligned} \tag{3}$$

with arbitrary functions J, K, L, M and N. In order to construct a BRST cohomology it is natural to demand nilpotency of Q_q and R_q, which is sufficient to determine [11] the coefficient functions in terms of the leading coefficients $A(h)$ and $J(h)$. From (2)

$$\begin{aligned}
&B(h) = A(h+2) - A(h) \\
&C(h) = A(h-2) - A(h) \\
&E(h) = A(h+2) - 2A(h) + A(h-2) \\
&F(h)A(h) = [h]_q
\end{aligned} \tag{4}$$

as emphasized in [10], it is a matter of dynamics as to which constraint function will be the most natural†. The special cases $A(h) = [h]_q$ (see [5]) and $A(h) \propto [h/2]_q$ or $A(h) \propto h$ (see [10]) have already been considered. Correspondingly from (3), the conditions are

$$\begin{aligned}
&K(h) = J(h+2) - J(h) \\
&M(h) = J(h-2) - J(h) \\
&L(h) = J(h-2) - 2J(h) + J(h+2) \\
&J(h)N(h) = -[h]_q.
\end{aligned} \tag{5}$$

Further algebraic structure can be imposed by demanding that $W = \{Q_q, R_q\}$ be the BRST *completion* of the Casimir operator

$$C_q = fe + [\tfrac{1}{2}h]_q[\tfrac{1}{2}h + 1]_q \tag{6}$$

of the algebra; that is the additional requirement

$$W = C_q + \text{ (ghost terms)} \tag{7}$$

† In the $q \to 1$ limit the standard construction with $A(h) \to h$ is required

is required. This condition (7) gives the additional constraint

$$A(h)J(h) = 2[\tfrac{1}{2}h]_q[\tfrac{1}{2}(h+2)]_q - [h]_q. \tag{8}$$

Coupled with suitably defined inner products, this additional structure allows [12] the algebraic analogue of the Hodge decomposition theorem to be established, and the cohomology to be related to invariants of the representations. Specifically these results obtain for unitary representations of the algebra which admit an adjoint operation (*) associated with an involutory anti-automorphism, that is, corresponding to a real form of the algebra. In the case of $U_q(sl(2))$ we have [13] three possibilities:

$$\begin{array}{llll} \text{(i)} & U_q(\mathrm{su}(2)) \approx U_q(\mathrm{so}(3))\ (q\in\mathbb{R}): & h^* = h, & e^* = f, \quad f^* = e \\ \text{(ii)} & U_q(\mathrm{su}(1,1))\ (q\in\mathbb{R}): & h^* = h, & e^* = -f, \quad f^* = -e \\ \text{(iii)} & U_q(\mathrm{sl}(2,\mathbb{R}))\ (|q|=1): & h^* = -h, & e^* = -e, \quad f^* = -f. \end{array} \tag{9}$$

For each of the cases (i) to (iii), the conjugation * is extended consistently to the ghost sector, such that Q_q and R_q are nilpotent (see (4) and (5)), their anticommutator is a BRST completion of the Casimir C_q (see (6)), and $R_q = Q_q{}^*$.

Note first that $A(h)$ and $J(h)$ can only be defined up to rescaling by a non-zero complex number λ, since any such redefinition may be absorbed by an automorphism of the algebra of c^0 and $\bar{c}^0$ (multiplication by λ and λ^{-1}, respectively). For cases (i) and (ii) the first two terms of (2) and (3) lead to $c^{\pm *} = \bar{c}^{\pm}$ and $c^{\pm *} = -\bar{c}^{\pm}$, respectively, while the structure of the third terms yields $c^{0*} = \bar{c}^0$ and $J(h) = A(h)^*$ after taking into account the rescaling freedom. For case (iii) the same considerations lead to $c^{\pm *} = -\bar{c}^{\mp}$, $c^{0*} = \bar{c}^0$ but now $J(h) = A(-h)^*$. Using these results and rewriting the right-hand side of (8), the final condition is

$$\begin{array}{ll} \text{(i), (ii)} & A(h)A(h)^* = (q+q^{-1})[\tfrac{1}{2}h]_q{}^2 \\ \text{(iii)} & A(h)A(-h)^* = (q+q^{-1})[\tfrac{1}{2}h]_q{}^2. \end{array} \tag{10}$$

Interestingly an acceptable solution for all three cases can be found, namely

$$A(h) = \sqrt{q+q^{-1}}\,\left[\tfrac{1}{2}h\right]_q \tag{11}$$

which is precisely the solution found independently by Dayi [10].

This solution is different from that of [5] because there no discussion of real forms or conjugation was given, and it was assumed that the anticommutator of Q_q and R_q gave *precisely* the Casimir invariant C_q of (6), with *no* ghost terms, that is

$$\{Q_q, R_q\} = C_q. \tag{12}$$

In fact there is a misprint in the solution (9) of [5] for R_q; the correct expression reads

$$\begin{aligned} R_q = f\bar{c}^- + e\bar{c}^+ + \left(\frac{q^{h/2}-q^{-h/2}}{q^{h/2}+q^{-h/2}}\right)\Bigg\{ & \left(\frac{q+q^{-1}}{q-q^{-1}}\right)\bar{c}^0 + \left(\frac{q^{\frac{h}{2}-1}-q^{-\frac{h}{2}+1}}{q^{\frac{h}{2}-1}+q^{-\frac{h}{2}+1}}\right)\bar{c}^0\bar{c}^+c^+ \\ & - \left(\frac{q^{\frac{h}{2}+1}-q^{-\frac{h}{2}-1}}{q^{\frac{h}{2}+1}+q^{-\frac{h}{2}-1}}\right)\bar{c}^0\bar{c}^-c^- \\ & - \frac{2(q^2-q^{-2})}{(q^{\frac{h}{2}-1}+q^{-\frac{h}{2}+1})(q^{\frac{h}{2}+1}+q^{-\frac{h}{2}-1})}\bar{c}^+\bar{c}^-\bar{c}^0c^+c^- \Bigg\} \end{aligned} \tag{13}$$

For the cohomology of Q_q, the conclusions of [5] remain valid, that for q a root of unity, the existence of indecomposable representations with *vanishing* Casimir leads to the possibility of non-singlet, non-gauge-invariant physical states at ghost number zero. With the results of the present letter, this analysis can be extended to arbitrary ghost number and to representations of the real forms of the deformed algebra.

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